Int. J. Solids Structures, 1968, Vol. 4, pp. 959 to 974. Pergamon Press. Printed in Great Britain

STATISTICAL THEORY OF DISLOCATIONS

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Abstract—The paper is devoted to a statistical derivation of the equations governing a continuous distribution of dislocations in a linear elastic medium. We begin with a system of infinitesimal Somigliana dislocations moving in an elastic medium in accordance with the laws of dynamics of discrete dislocations. By introducing the classical phase space with its Liouville and transport equations and defining the appropriate expectation values we derive in the usual manner the equations for the density of the "dislocation fluid", its velocity and the average elastic field. As a result we arrive at a compound continuous medium D_R constituting a mixture of a material elastic body and the dislocation fluid. The system of equations constitutes a system of seven quasi-linear partial differential equations which are shown to be hyperbolic under certain definite conditions. Some general features of the system are discussed and a one-dimensional example examined in more detail to demonstrate some properties of the D_R medium; thus, shock waves and slip planes are shown to exist. The possibility of constructing in this manner "plastic" or "elastic–plastic" media is briefly considered.

INTRODUCTION

THIS paper is devoted to a statistical derivation of the equations governing a continuous distribution of dislocations in a linear elastic medium. We begin with a system of a finite number of infinitesimal Somigliana dislocations moving in a linear elastic medium in accordance with the laws of dynamics of discrete dislocations [1]; a change of the model of the defect will not influence the general procedure we employ, as long as the defect is infinitesimal (see e.g. Appendix A for the derivation of the equations governing continuous distributions of vacancies). By introducing the classical phase space with its Liouville and transport equations and defining the expectation values of various physical quantities connected with the motion of the system, we introduce the Kirkwood formalism making an extensive use of the Dirac delta functions [2, 3]. This formalism excellently serves our purpose, since the delta functions appear initially in our problem in the right-hand sides of the Lamé equations, expressing the influence of the dislocations on the generation of the elastic displacement field. By ordinary means, as in statistical hydrodynamics, on the basis of the transport equation we now derive the equations for the density $\nu(\mathbf{x}, t)$ and the velocity $\mathbf{v}(\mathbf{x}, t)$ of the dislocation fluid—the continuity and linear momentum equations; the influence of the displacement (or rather the stresses and the velocity) of the elastic body is expressed by certain terms in the equation of conservation of the linear momentum. Further, we introduce the expectation value of the displacement of the elastic body, the displacement being a random function since it depends on the distribution and motion of the dislocations by which it is produced; this makes it possible to perform the averaging procedure on the Lamé equations and to derive an equation for the expectation value of the displacement. The procedure employed here is similar to that used in [4] in an investigation of the macroscopic Maxwell field. Incidentally, the averaging of the Lamé equations results in a very natural definition of quantities describing the continuous distribution of dislocations, such as the dislocation density tensor, introduced earlier by other authors [5].

The analogy to the statistical derivation of the hydrodynamics equations is evident; as a result we obtain a compound continuous medium D_R constituting a mixture (in the sense that each geometric point is occupied by two particles, [6, 7]; in our case one of the particles is material, the other may not be material) of the material elastic body and the dislocation fluid. We do not carry out an analysis of the derived system of seven quasi-linear partial differential equations, confining ourselves to some general remarks and an example of a one-dimensional motion of tangential dislocations, for only in the case of one spatial variable a fairly complete theory of such equations exists. We believe that this example illustrates the basic features of the D_R medium, such as the creation and propagation of "slip planes" and "shock waves", various types of waves, etc., and indicates the relation of the D_R medium to the classical "elastic–plastic" body. Obviously, only a further investigation, first of all thermodynamic considerations, can decide whether it is possible to construct by statistical methods on the basis of the dynamics of discrete dislocations or other defects, a rational theory of plasticity. There are strong indications that the answer is positive.

The structure of the D_R medium investigated in this paper is restricted in the following sense. In the general case the dislocations are characterized by the vectorial intensity $\underset{\alpha}{\mathbf{U}}$ and the director $\underset{\alpha}{\mathbf{n}}$, different for each dislocation; the general theory constructed under this assumption is rather complicated, the fundamental system of differential equations containing forty-two equations with the following unknowns: the displacement vector $\mathbf{u}$, the velocity of the dislocation fluid $\mathbf{v}$, the dislocation density tensor $\boldsymbol{\kappa}$ and the dislocation velocity tensor $\boldsymbol{\varepsilon}$; moreover, instead of one constitutive equation at least three are required. Needless to say, the complexity of the system of equations makes any conclusions or practical applications almost impossible; this is emphasized by the fact that presently very little is known about the properties of the dislocation fluid and no basis is known for establishing the required constitutive relations. We assume therefore that $\underset{\alpha}{\mathbf{U}} = \mathbf{U}$ and $\underset{\alpha}{\mathbf{n}} = \mathbf{n}$, i.e. these two vectors are the same for all dislocations. Then $\boldsymbol{\kappa} = \mathbf{U}\mathbf{n}\nu$ and $\boldsymbol{\epsilon} = \mathbf{U}\mathbf{n}\mathbf{v}\nu$. In other words we endow the medium with a homogeneous structure $\{\mathbf{U}, \mathbf{n}\}$ and seek only the density and velocity of the dislocation fluid. A generalization to a mixture of several such fluids is straightforward.

We do not attempt at this stage to compare our theory with the existing theories of continuous distribution of dislocations [8–11]. First of all the basic model of the defect is different; secondly our fundamental variables are different. In fact, we introduce from the very beginning a displacement function, the existence of which in most of the theories is denied. It seems that only a comparison of the final results will be possible.

1. DYNAMICS AND STATISTICS OF DISCRETE DEFECTS

In this section we state the basic relations and formulae concerning the motion of discrete dislocations ([12], [1]); further, we introduce the appropriate phase space and write down the Liouville and transport equations. The latter constitutes the basis of the derivation of the fundamental equations of the continuum theory of dislocations by means of the Kirkwood formalism.

The equation of motion of a single dislocation (infinitesimal Somigliana dislocation) "α" in an elastic field produced by external sources "0" and other dislocations "β" has the form

$$\underset{\alpha}{\mathbf{\dot{p}}} = \underset{\alpha}{\mathbf{f}_T} = \underset{\alpha 0}{\mathbf{f}} + \sum_{\substack{\beta \\ \beta \neq \alpha}} \underset{\alpha\beta}{\mathbf{f}}. \tag{1.1}$$

Here $\mathbf{p}$ is the linear momentum of the dislocation; if we neglect terms of the order $\underset{\alpha}{v}/c_2^2$ as compared with unity (linearized theory) $\underset{\alpha}{\mathbf{p}}$ is given in terms of the velocity $\underset{\alpha}{\mathbf{v}}$ by the relation

$$\underset{\alpha}{p^i} = -\underset{\alpha}{n^{ip}}\underset{\alpha}{\ddot{v}_p} + \underset{\alpha}{m^{ip}}\underset{\alpha}{v_p}. \tag{1.2}$$

It was shown in [1] that the first term in (1.2) is proportional to a small parameter t_0^2 where t_0 is the time required for a sound signal to travel across the dislocation surface; it will be neglected in this paper.† We have therefore

$$\underset{\alpha}{p^i} = \underset{\alpha}{m^{ip}}\underset{\alpha}{v_p} \tag{1.3}$$

where the tensorial mass $\underset{\alpha}{m^{ip}}$ is constant in time and given by the formula‡

$$\underset{\alpha}{m^{ip}} = \mu c_2^{-5}\Delta^1[\delta^{ip}(m_1 \underset{\alpha}{U^2} + m_2 \underset{\alpha}{U^2_{(n)}}) + m_3 \underset{\alpha}{U_{(n)}}\underset{\alpha}{n^{(i}}\underset{\alpha}{U^{p)}} + m_4 \underset{\alpha}{U^i}\underset{\alpha}{U^p} + m_5 \underset{\alpha}{U^2}\underset{\alpha}{n^i}\underset{\alpha}{n^p}]. \tag{1.4}$$

The tensor $\underset{\alpha}{m^{ip}}$ is symmetric, non-singular. Its inverse will be denoted by $\underset{\alpha}{\overset{-1}{m}{}^{ip}}$.

Here μ is the Lamé constant, c_2 the second sound velocity, Δ^1 is an infinite integral of the order t_0^{-3} assumed here to be a finite undetermined constant, $m_1, \ldots, m_5$ are numerical coefficients depending only on the Poisson ratio. Finally $\underset{\alpha}{U^i} = \underset{\alpha}{b^i}\underset{\alpha}{s}$, where $\underset{\alpha}{b^i}$ is the Burgers vector and $\underset{\alpha}{s}$ is the surface of the defect, is the displacement discontinuity vector and $\underset{\alpha}{n_i}$ is the unit vector normal to the surface.

The force exerted by the external field $\underset{0}{\mathbf{u}}(\mathbf{x}, t)$ on the dislocation α is given by the formula

$$\underset{\alpha 0}{f_i}(t) = -\mu \underset{\alpha}{\varkappa_{pq}}\sigma^{pqrs}\nabla_i\nabla_r \underset{0}{u_s} + \rho \underset{\alpha}{\varkappa_{pi}}\frac{\partial^2 \underset{0}{u^p}}{\partial t^2} - 2\rho \underset{\alpha}{\varkappa_{r[p}}\underset{\alpha}{v^p}\nabla_{i]}\frac{\partial \underset{0}{u^r}}{\partial t} \tag{1.5}$$

where $\underset{\alpha}{\varkappa_{ij}} = \underset{\alpha}{U_i}\underset{\alpha}{n_j}$. Similarly, for the force of dislocation β on dislocation α we have

$$\underset{\alpha\beta}{f_i}(t) = -\mu \underset{\alpha}{\varkappa_{pq}}\sigma^{pqrs}\nabla_i\nabla_r \underset{\beta}{u_s} + \rho \underset{\alpha}{\varkappa_{pi}}\frac{\partial^2 \underset{\beta}{u^p}}{\partial t^2} - 2\rho \underset{\alpha}{\varkappa_{r[p}}\underset{\alpha}{v^p}\nabla_{i]}\frac{\partial \underset{\beta}{u^r}}{\partial t}. \tag{1.6}$$

Here $\sigma^{pqrs} = (\lambda/\mu)\delta^{pq}\delta^{rs} + 2\delta^{p(r}\delta^{q)s}$. Observe finally that since $\underset{\alpha\beta}{f_i}$ given formally by (1.6) in which we set $\alpha = \beta$, is given by the relation [5]

$$\underset{\alpha\alpha}{f_i} = -\tfrac{1}{2}\underset{\alpha}{\dot{p}_i} \tag{1.7}$$

the equation of motion (1.1) can be written in the form

$$\underset{\alpha}{\mathbf{\dot{p}}} = 2(\underset{\alpha 0}{\mathbf{f}} + \sum_{\beta} \underset{\alpha\beta}{\mathbf{f}}). \tag{1.8}$$

This is the form we shall use below.

† A generalization presents no difficulties; it leads to a system of parabolic rather than hyperbolic equations.

‡ For the case of dislocations possessing real mass see Appendix B.

The dynamics of a system of dislocations is contained in equations (1.1)–(1.11) and the energy conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}(\underset{\alpha}{m}^{pq}\underset{\alpha}{v}_p\underset{\alpha}{v}_q+\underset{\alpha}{n}^{pq}\underset{\alpha}{\dot{v}}_p\underset{\alpha}{\dot{v}}_q)-\underset{\alpha}{n}^{pq}\underset{\alpha}{v}_p\underset{\alpha}{\ddot{v}}_q] = 0 \tag{1.9}$$

i.e. neglecting $\underset{\alpha}{n}^{pq}$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}\underset{\alpha}{m}^{pq}\underset{\alpha}{v}_p\underset{\alpha}{v}_q) = 0. \tag{1.10}$$

The quantity $\frac{1}{2}\underset{\alpha}{m}^{pq}\underset{\alpha}{v}_p\underset{\alpha}{v}_q$ is naturally called the kinetic energy of the dislocation. Now we are in a position to proceed to the statistical concepts.

Our phase space Γ, just as in the case of statistics of a system of material particles obeying the laws of Newtonian mechanics, is a $6N$-dimensional space representing $3N$ coordinates $\underset{\alpha}{\xi}$ and $3N$ linear momenta $\underset{\alpha}{\mathbf{p}}$ of the dislocations. The probability distribution function will be denoted by

$$f = f(\underset{\alpha}{\xi}, \underset{\alpha}{\mathbf{p}}, t) \tag{1.11}$$

and subjected to the normalization condition

$$\int f\,\mathrm{d}\Gamma = 1, \qquad \mathrm{d}\Gamma = \mathrm{d}\underset{1}{\xi}, \ldots, \mathrm{d}\underset{N}{\xi}\,\mathrm{d}\underset{1}{\mathbf{p}}, \ldots, \mathrm{d}\underset{N}{\mathbf{p}}. \tag{1.12}$$

It satisfies the Liouville equation

$$\frac{\partial f}{\partial t}+\sum_{\alpha}\left(\overset{-1}{\underset{\alpha}{m}}{}^{pq}\underset{\alpha}{p}_p\frac{\partial f}{\partial \underset{\alpha}{\xi}{}^q}+\underset{\alpha}{f}{}^p_T\frac{\partial f}{\partial \underset{\alpha}{p}{}^p}\right) = 0 \tag{1.13}$$

For any dynamical variable which may in addition to $\underset{\alpha}{\xi}$, $\underset{\alpha}{\mathbf{p}}$, t depend on the spatial Eulerian coordinates $\mathbf{x}$, $\mathbf{P} = \mathbf{P}(\underset{\alpha}{\xi}, \underset{\alpha}{\mathbf{p}}, \mathbf{x}, t)$ we have the transport equation

$$\frac{\partial}{\partial t}\langle P\rangle = \left\langle\frac{\mathrm{d}P}{\mathrm{d}t}\right\rangle \tag{1.14}$$

where $\langle P\rangle$ is the expectation value of P, i.e.

$$\langle P\rangle = \int Pf\,\mathrm{d}\Gamma. \tag{1.15}$$

The transport equation (1.14) will be used in the derivation of the equations of the continuous dislocation fluid in an elastic body; this compound medium will be called the D medium.

2. THE FIELD EQUATIONS

We begin the construction of the set of equations describing the D medium by deriving the displacement equations. Introduce first the total average displacement by the formula

$$\overset{\mathrm{av}}{\mathbf{u}}(\mathbf{x}, t) = \langle\underset{0}{\mathbf{u}}(\mathbf{x}, t)+\sum_{\alpha}\underset{\alpha}{\mathbf{u}}(\mathbf{x}, t)\rangle = \underset{0}{\mathbf{u}}(\mathbf{x}, t)+\sum_{\alpha}\langle\underset{\alpha}{\mathbf{u}}(\mathbf{x}, t)\rangle \tag{2.1}$$

where $\underset{\alpha}{\mathbf{u}}(\mathbf{x}, t)$ is the displacement produced by the dislocation α and $\underset{0}{\mathbf{u}}(\mathbf{x}, t)$ is due to the body or external forces. We have [12],

$$\mathbf{L}\underset{0}{\mathbf{u}} = \mathbf{X}, \tag{2.2}$$

$$\mathbf{L}\underset{\alpha}{\mathbf{u}}_i = -\int_{-\infty}^{t} \mathrm{d}\tau \underset{\alpha}{U}_j(\mu \underset{\alpha}{n}_m \sigma^{jmp}{}_i \nabla_p + \rho \underset{\alpha}{v}_{(n)} \delta_i^j)\delta(\underset{\alpha}{\xi}_\tau - \mathbf{x})\delta(t-\tau) \tag{2.3}$$

where

$$\mathbf{L} = L_{ij} = \rho\delta_{ij}\frac{\partial^2}{\partial t^2} - \mu\delta_{ij}\nabla - (\lambda+\mu)\nabla_i\nabla_j\,,$$

or, performing the integration with respect to time we obtain for $\mathbf{u} = \underset{0}{\mathbf{u}} + \sum_\alpha \underset{\alpha}{\mathbf{u}}$

$$Lu_i = X_i - \mu \sum_\alpha [\underset{\alpha}{U}_j(\underset{m}{n}\sigma^{jmp}{}_i - c_2^{-2}\delta_i^j \underset{\alpha}{v}_{(n)} \underset{\alpha}{v}^p)\nabla_p \delta(\underset{\alpha}{\xi} - \mathbf{x}) - c_2^{-2} \underset{\alpha}{U}_i \underset{\alpha}{v}_{(n)} \delta(\underset{\alpha}{\xi} - \mathbf{x})]. \tag{2.4}$$

This is an equation for the total displacement field produced by the body forces and dislocations; we now proceed to deduce on the basis of (2.4) an equation for $\overset{\mathrm{av}}{\mathbf{u}}(\mathbf{x}, t)$. To this end we multiply (2.4) by f and integrate over the phase space. Since f is independent of the point in the physical space, the operator $\mathbf{L}$ commutes with $\langle\ \rangle$; we have

$$\langle Lu_i \rangle = L\overset{\mathrm{av}}{u}_i \tag{2.5}$$

and obviously

$$\langle X_i \rangle = X_i. \tag{2.6}$$

To investigate the right-hand side of (2.4) we first introduce the following quantities:
(i) the dislocation density

$$\nu(\mathbf{x}, t) = \sum_\alpha \langle \delta(\underset{\alpha}{\xi} - \mathbf{x}) \rangle = \sum_\alpha \langle \underset{\alpha}{\delta} \rangle. \tag{2.7}$$

(ii) The dislocation velocity

$$\nu(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t) = \sum_\alpha \langle \underset{\alpha}{\mathbf{v}}\underset{\alpha}{\delta} \rangle. \tag{2.8}$$

(iii) The dislocation density tensor

$$\boldsymbol{\kappa}(\mathbf{x}, t) = \sum_\alpha \langle \underset{\alpha}{\mathbf{U}}\underset{\alpha}{\mathbf{n}}\underset{\alpha}{\delta} \rangle. \tag{2.9}$$

(iv) The dislocation velocity tensor

$$\boldsymbol{\epsilon}(\mathbf{x}, t) = \sum_\alpha \langle \underset{\alpha}{\mathbf{U}}\underset{\alpha}{\mathbf{n}}\underset{\alpha}{\mathbf{v}}\underset{\alpha}{\delta} \rangle. \tag{2.10}$$

Taking into account that

$$\sum_\alpha \langle \underset{\alpha}{U_i} \underset{\alpha}{n_q} \underset{\alpha}{v^p} \underset{\alpha}{v^q} \underset{\alpha}{\delta} \rangle = \sum_\alpha \left\langle \left(\underset{\alpha}{U_i} \underset{\alpha}{n_q} \underset{\alpha}{v^p} - \frac{1}{v} \epsilon_{iq}{}^p \right) (\underset{\alpha}{v^q} - v^q) \underset{\alpha}{\delta} \right\rangle + \epsilon_{iq}{}^p v^q$$

$$\sum_\alpha \left\langle \frac{\mathrm{d}}{\mathrm{d}t} (\underset{\alpha}{U_i} \underset{\alpha}{n_p} \underset{\alpha}{v^p}) \underset{\alpha}{\delta} \right\rangle = \sum_\alpha \left\langle \frac{\mathrm{d}}{\mathrm{d}t} (\underset{\alpha}{U_i} \underset{\alpha}{n_p} \underset{\alpha}{v^p} \underset{\alpha}{\delta}) \right\rangle + \sum_\alpha \nabla_q \langle \underset{\alpha}{U_i} \underset{\alpha}{n_p} \underset{\alpha}{v^p} \underset{\alpha}{v^q} \underset{\alpha}{\delta} \rangle$$

$$= \frac{\partial}{\partial t} \epsilon_{ip}{}^p + \nabla_q \left[\sum_\alpha \left\langle \left(\underset{\alpha}{U_i} \underset{\alpha}{n_p} \underset{\alpha}{v^q} - \frac{1}{v} \epsilon_{ip}{}^q \right) (\underset{\alpha}{v^p} - v^p) \underset{\alpha}{\delta} \right\rangle + \epsilon_{ip}{}^q v^p \right]$$

after simple transformations we find that the averaged equation (2.4) takes the form (we drop the "av" over the displacement)

$$Lu_i = X_i + \mu \left(\sigma^{pqr}{}_i \nabla_r \varkappa_{pq} + c_2^{-2} \frac{\partial}{\partial t} \epsilon_{ip}{}^p \right). \tag{2.11}$$

Observe that

$$\sigma^{pqr}{}_i \nabla_r \varkappa_{pq} = \frac{1}{\mu} \nabla_p \underset{D}{\sigma_i}{}^p \tag{2.12}$$

where $\underset{D}{\sigma_{ip}} = \lambda \delta_{ip} \varkappa_q{}^q + 2\mu \varkappa_{(ip)}$ may be called "the stress tensor due to the strain $\varkappa_{(ij)}$". Consequently, denoting by $\underset{H}{\sigma^{ij}}$ the Hookean stress based on the displacement u_i, equation (2.12) may be written in the form

$$\nabla_p (\underset{H}{\sigma^{ip}} + \underset{D}{\sigma^{ip}}) - \rho \frac{\partial}{\partial t} \left(\frac{\partial u^i}{\partial t} - \epsilon^{ip}{}_p \right) = -X^i. \tag{2.11$'$}$$

In other words, the influence of the dislocation fluid is expressed by the change of the stress tensor by $\underset{D}{\sigma^{ij}}$ and a change of the inertia force by $-\rho(\partial/\partial t)\epsilon^{ip}{}_p$.

The system of the field equations (2.11), therefore, contains two unknown tensors describing the distribution and motion of the dislocations; in order to obtain a full set of equations for the D medium we have to derive the equations describing $\boldsymbol{\kappa}$ and $\boldsymbol{\epsilon}$; this can be done by means of the transport equation (1.14). Prior to that, however, for the reasons stated in Introduction we assume that the intensity and director of the dislocation are independent of α, i.e.

$$\underset{\alpha}{U_i} = U_i, \qquad \underset{\alpha}{n_i} = n_i. \tag{2.13}$$

We recall, that both U_i and n_i are constant in time. We are now dealing with an elastic body filled with identical dislocations. This medium will be denoted by D_R.

Under the assumption (2.13) we have

$$\boldsymbol{\kappa}(\mathbf{x}, t) = \mathbf{Un}v(\mathbf{x}, t); \qquad \boldsymbol{\epsilon}(\mathbf{x}, t) = \mathbf{Unv}(\mathbf{x}, t)v(\mathbf{x}, t) \tag{2.14}$$

and the field equations (2.11) take the form (we set $\varkappa_{ij} = U_i n_j$)

$$Lu_i - \mu\left[\varkappa_{pq}\sigma^{pqr}{}_i\nabla_r \nu + c_2^{-2}\varkappa_{ip}\frac{\partial}{\partial t}(\nu v^p)\right] = 0. \tag{2.15}$$

The two unknown quantities are now $\nu(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$. We proceed to deduce the equations governing these fields.

To derive the equation for $\nu(\mathbf{x}, t)$ we set in the transport equation (1.14) $P = \sum_\alpha \delta_\alpha$; after standard calculations (see e.g. [2]) we obtain for $\nu(\mathbf{x}, t)$ the ordinary continuity equation

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \mathbf{v}) = 0. \tag{2.16}$$

The equation for $\mathbf{v}(\mathbf{x}, t)$ is the equation of conservation of linear momentum; we set here

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha \delta_\alpha. \tag{2.17}$$

Again, simple calculations yield the equation

$$\frac{\partial}{\partial t}\mathbf{p} + \nabla \cdot (\mathbf{v}\mathbf{p}) = \nabla \cdot \underset{K}{\boldsymbol{\sigma}} + \sum_\alpha \langle \dot{\mathbf{p}}_\alpha \delta_\alpha \rangle \tag{2.18}$$

where

$$\mathbf{p}(\mathbf{x}, t) = \sum_\alpha \langle \mathbf{p}_\alpha \delta_\alpha \rangle. \tag{2.19}$$

In our case $\mathbf{p}_\alpha = \mathbf{m} \cdot \mathbf{v}_\alpha$ and therefore

$$\mathbf{p} = \nu \mathbf{m} \cdot \mathbf{v}. \tag{2.20}$$

Further,

$$\underset{K}{\boldsymbol{\sigma}}(\mathbf{x}, t) = -\sum_\alpha \left\langle (\mathbf{v}_\alpha - \mathbf{v})\left(\mathbf{p}_\alpha - \frac{1}{\nu}\mathbf{p}\right)\delta_\alpha \right\rangle \tag{2.21}$$

or

$$\underset{K}{\boldsymbol{\sigma}}^{ij}(\mathbf{x}, t) = -m^{jp}\sum_\alpha \langle (v_\alpha^i - v^i)(v_{\alpha p} - v_p)\delta_\alpha \rangle. \tag{2.22}$$

By analogy to the theory of fluids $\underset{K}{\boldsymbol{\sigma}}$ will be called the kinetic stress tensor; as well known a system of phenomenological equations derived by means of statistical methods is never closed and we require for the stress a constitutive relation. It remains to calculate the last term in (2.18); to this end we use the equation of motion of a single dislocation (1.1). Thus,

$$\sum_\alpha \langle \dot{\mathbf{p}}_\alpha \delta_\alpha \rangle = 2\Big(\sum_\alpha \langle \mathbf{f}_{\alpha 0} \delta_\alpha \rangle + \sum_{\alpha,\beta} \langle \mathbf{f}_{\alpha\beta} \delta_\alpha \rangle\Big). \tag{2.23}$$

There is no difficulty with the first term, as shown below. The second term is due to the interaction between the dislocations; in deriving the hydrodynamics equations where the interaction is given in terms of potential energy depending on the positions of the particles, this term is dealt with as follows: on the basis of the properties of the potential energy it is proved that this expression is a divergence of a tensor called the stress tensor and a constitutive relation is set up for the sum of the latter and $\underset{K}{\boldsymbol{\sigma}}$. As well known, in dilute gases $\underset{K}{\boldsymbol{\sigma}}$ is the dominant term while in dense liquids it is small as compared with the interaction stress tensor. In our case this procedure is not quite satisfactory. First observe that since $\underset{\alpha\beta}{\mathbf{f}} + \underset{\beta\alpha}{\mathbf{f}} \neq 0$ ([12]) the considered term is not a divergence of a tensor; of course we could try to establish a constitutive relation for the term $\sum_{\alpha,\beta} \langle \underset{\alpha\beta}{\mathbf{f}} \underset{\alpha}{\delta} \rangle$ itself, in view, however, of the scarcity of experimental data concerning continuous distributions of dislocations this is hardly possible. Further, according to (1.9) the forces $\underset{\alpha\beta}{\mathbf{f}}$ are given in terms of the fields $\underset{\alpha}{\mathbf{u}}(\underset{\beta}{\boldsymbol{\xi}}, t)$ through which the dislocations interact; therefore, we shall attempt to express the term $\sum_{\alpha} \langle \underset{\alpha}{\dot{\mathbf{p}}} \underset{\alpha}{\delta} \rangle$ by the average displacement field $\mathbf{u}(\mathbf{x}, t)$. Of course, it turns out that an approximation is required here, we believe, however, that its introduction does not influence essentially the properties of the D_R medium.

Consider first the term due to the external field; making use of (1.5), (2.7), (1.9) and (2.14) we obtain

$$\sum_{\alpha} \langle \underset{\alpha 0}{f_i} \underset{\alpha}{\delta} \rangle = -\mu\nu \left(\varkappa_{pq}\sigma^{pqrs}\nabla_i\nabla_r \underset{0}{u_s} - c_2^{-2}\varkappa_{pi}\frac{\partial^2 \underset{0}{u^p}}{\partial t^2} - 2c_2^{-2}\varkappa_{r[i}v^p\nabla_{p]}\frac{\partial \underset{0}{u^r}}{\partial t} \right). \tag{2.24}$$

Let us now proceed to the interaction; we first examine the first term of $\underset{\alpha\beta}{f_i}$ given by (1.9). Making use of equation (2.1)

$$\sum_{\alpha,\beta} \langle \nabla_i\nabla_r \underset{\beta}{u_s} \underset{\alpha}{\delta} \rangle = \sum_{\alpha,\beta} \langle (\nabla_i\nabla_r \underset{\beta}{u_s} - \langle \nabla_i\nabla_r \underset{\beta}{u_s} \rangle) \underset{\alpha}{\delta} \rangle + \nu\nabla_i\nabla_r(u_s - \underset{0}{u_s}). \tag{2.25}$$

Transforming in a similar manner the remaining terms of the force $\underset{\alpha\beta}{f_i}$ and substituting into (2.23) we have

$$\sum_{\alpha} \langle \underset{\alpha}{\dot{p}_i} \underset{\alpha}{\delta} \rangle = -\mu\nu \left(\varkappa_{pq}\sigma^{pqrs}\nabla_i\nabla_r u_s - c_2^{-2}\varkappa_{pi}\frac{\partial^2 u^p}{\partial t^2} - 2c_2^{-2}\varkappa_{r[i}v^p\nabla_{p]}\frac{\partial u^r}{\partial t} \right) + F_i \tag{2.26}$$

where

$$\begin{aligned} F_i(\mathbf{x}, t) = -\mu \Bigg[& \varkappa_{pq}\sigma^{pqrs} \sum_{\alpha,\beta} \langle (\nabla_i\nabla_r \underset{\beta}{u_s} - \langle \nabla_i\nabla_r \underset{\beta}{u_s} \rangle) \underset{\alpha}{\delta} \rangle \\ & - c_2^{-2}\varkappa_{pi} \sum_{\alpha,\beta} \left\langle \left(\frac{\partial^2 \underset{\beta}{u^p}}{\partial t^2} - \left\langle \frac{\partial^2 \underset{\beta}{u^p}}{\partial t^2} \right\rangle \right) \underset{\alpha}{\delta} \right\rangle \\ & - 2c_2^{-2}\varkappa_{r[p} \sum_{\alpha,\beta} \left\langle \left(\nabla_{i]}\frac{\partial \underset{\beta}{u^r}}{\partial t} - \left\langle \nabla_{i]}\frac{\partial \underset{\beta}{u^r}}{\partial t} \right\rangle \right) \underset{\alpha}{v^p} \underset{\alpha}{\delta} \right\rangle \Bigg]. \end{aligned} \tag{2.27}$$

In the case of instantaneous interactions, i.e. when in the expression for $\underset{\alpha\beta}{\mathbf{f}}$ (see (1.6) and the formula for $\underset{\beta}{\mathbf{u}}$ in [1]) retardations in the transfer of the action are neglected, it can be proved that $\mathbf{F}(\mathbf{x}, t)$ is a divergence of a tensor, i.e. $F_i = \nabla_p \underset{P}{\sigma_i^p}$; then we set $\underset{K}{\boldsymbol{\sigma}} + \underset{P}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$ is the only quantity for which a constitutive relation is required. We assume hereafter that this is the case. There are also reasons for neglecting **F** altogether, at least in the first approximation, on the following basis. The typical transformation (2.25) aims at replacing the average of a product by the product of the averages; the difference of these two $\sum_{\alpha,\beta} \langle(\nabla_i \nabla_r \underset{\beta}{u_s} - \langle\nabla_i \nabla_r \underset{\beta}{u_s}\rangle)\underset{\alpha}{\delta}\rangle = \sum_{\alpha,\beta} \langle(\nabla_i \nabla_r \underset{\beta}{u_s} - \langle\nabla_i \nabla_r \underset{\beta}{u_s}\rangle)(\underset{\alpha}{\delta} - \langle\underset{\alpha}{\delta}\rangle)\rangle$ constitutes the correlation between the considered quantities. It can readily be proved that the integrand in this expression is proportional to the difference $\overset{2}{f}_{\alpha\beta} - \overset{1}{f}_{\alpha}\overset{1}{f}_{\beta}$ where $\overset{2}{f}_{\alpha\beta}$ is the binary distribution function and $\overset{1}{f}_{\alpha}$ is that for a single particle (see e.g. [13]). Thus, $F_i(x, t) \equiv 0$ when $\overset{2}{f}_{\alpha\beta} = \overset{1}{f}_{\alpha}\overset{1}{f}_{\beta}$. Our assumption, therefore, means that the interaction between dislocations is completely described by the term in parenthesis in (2.26). The same can be expressed in a somewhat different form; since the considered expressions are proportional to the differences between the current values and the corresponding averages, we assume that the fluctuations are vanishingly small as compared with the average values themselves. A further discussion of this problem requires a more detailed examination of the distribution function (kinetic equations, etc.) and the structure of the dislocation fluid.

We are now in a position to write down the final form of the linear momentum equation (2.18); prior to that, however, let us mention the following fact. It will be proved in deriving the principle of conservation of energy that in the continuous exchange of energy between the elastic body and the dislocation fluid, the contribution of the last term in parenthesis in (2.26) is of the order of v^2/c_2^2 as compared with the contribution of the last term in (2.15); since we assumed in the dynamics of discrete dislocations that $\underset{\alpha}{v}^2/c_2^2 \ll 1$ and the average velocity cannot be greater than that of separate dislocations, we may neglect the considered term on the basis that its energy contribution is negligible. This is done for consistence only; it does simplify somewhat the equation, but does not change any of its basic properties. Thus, after simple transformations, making use of the continuity equation (2.16) we finally obtain

$$\frac{Dv^i}{Dt} - \frac{1}{\nu} \overset{-1}{m}{}^i{}_n \nabla_p \sigma^{np} + 2\mu \overset{-1}{m}{}^{in} \left(\varkappa_{pq} \sigma^{pqrs} \nabla_n \nabla_r u_s - c_2^{-2} \varkappa_{pn} \frac{\partial^2 u^p}{\partial t^2} \right) = 0 \tag{2.28}$$

or

$$\frac{Dv^i}{Dt} - \frac{1}{\nu} \overset{-1}{m}{}^i{}_n \nabla_p \sigma^{np} + 2\mu \overset{-1}{m}{}^{in} \varkappa_{pq} \left(\nabla_n \underset{H}{\sigma}{}^{pq} - \rho \delta_n^q \frac{\partial^2 u^p}{\partial t^2} \right) = 0 \tag{2.28'}$$

where $\underset{H}{\sigma}{}^{ij}$ is the Hookean stress tensor based on the displacement u_i.

The first two terms in (2.28) are the same as in the hydrodynamics of a perfect fluid; the term

$$2 \overset{-1}{m}{}^{in} \varkappa_{pq} \left(\nabla_n \underset{H}{\sigma}{}^{pq} - \rho \delta_n^q \frac{\partial^2 u^p}{\partial t^2} \right)$$

expresses the influence of the displacement field in the elastic body on the motion of the dislocation fluid.

If we postulate a constitutive relation for σ^{ij} the system of equations (2.12), (2.16) and (2.28) constitutes a system of seven equations with seven unknowns $\mathbf{u}(\mathbf{x}, t)$, $\nu(\mathbf{x}, t)$ and

$\mathbf{v}(\mathbf{x}, t)$. As mentioned before, there is at present very little experimental data for establishing the constitutive relation, in this paper therefore, we make the simplest assumption that the fluid is perfect and barotropic, i.e. that (cf. equation (2.22)).

$$\sigma^{ij} = -m^{ij}p(\nu). \tag{2.29}$$

Furthermore, assuming that the flow is "adiabatic" we set

$$p(\nu) = p_0\left(\frac{\nu}{\nu_0}\right)^{\gamma} = \alpha\nu^{\gamma} \tag{2.30}$$

where γ, p_0, ν_0 are constants; γ will be called the adiabatic exponent. As in the theory of gases (2.29) and (2.30) can to a certain extent be justified by means of the kinetic and thermodynamic argument, we regard it here, however, as a postulate. Denoting $c^2 = \mathrm{d}p/\mathrm{d}\nu$ which will hereafter be called the propagation velocity, we finally arrive at the following system of equations:

$$\begin{aligned}
&\mu\nabla^2 u_i + (\lambda+\mu)\nabla_i\nabla_p u^p - \rho\frac{\partial^2 u_i}{\partial t^2} + \mu\varkappa_{pq}\left[\sigma^{pqr}{}_i\nabla_r\nu + c_2^{-2}\delta_i^p\frac{\partial}{\partial t}(\nu v^q)\right] = -X_i \\
&\frac{\partial \nu}{\partial t} + \nabla_p(\nu v^p) = 0 \\
&\frac{Dv_i}{Dt} + \frac{c^2}{\nu}\nabla_i\nu + 2\mu \overset{-1}{m}{}_i{}^n\varkappa_{pq}\left(\sigma^{pqrs}\nabla_n\nabla_r u_s - c_2^{-2}\delta_n^q\frac{\partial^2 u^p}{\partial t^2}\right) = 0.
\end{aligned} \tag{2.31}$$

This is the required system of seven quasi-linear equations with seven unknowns, describing the D_R medium. The system contains second derivatives of $\mathbf{u}$ and first derivatives of ν and $\mathbf{v}$; by ordinary substitutions it can readily be reduced to a system of nineteen first order quasi-linear partial differential equations. A fairly complete theory of such systems exists only in the case of one spatial variable; in the general case only local existence theorems and uniqueness for regular solutions can be proved (see e.g. [14]). It is well known from the theory of one-dimensional problems ([14–17]) that the general regular solutions exist only locally, i.e. independently of the smoothness of the initial data there arise after a finite time discontinuity surfaces, which in our case are shock waves and slip planes, propagated with velocities different from characteristic. Another important property of our systems concerns the irreversibility of the process it describes; the situation here is the following. The system of equations (2.31) is invariant with respect to the inversion of time; this statement however, has a meaning only in the case of regular solutions when all functions $\mathbf{u}$, ν, $\mathbf{v}$, $\nabla\mathbf{u}$, $\partial\mathbf{u}/\partial t$ are continuous. If, now, we are faced with discontinuous (generalized, weak) solutions which is in fact the case, we have the following results ([15, 16]); if we start with a generalized stable solution of the Cauchy problem denoting it by $\mathbf{A}(\mathbf{x}, t)$ with the initial conditions at $t = 0$, then the Cauchy problem posed for $t < t_1$ with the initial values $\mathbf{A}(\mathbf{x}, t_1)$ at t_1 is unstable. This fact is due to the generation of the discontinuity surfaces and in gas dynamics expresses the phenomenon of increase of entropy in passing through the shock wave. In this paper the thermodynamics is not considered at all, no entropy has been introduced and the principle of conservation of energy is derived on the basis of the phenomenological equations (2.31) rather than from the theory of discrete system via the transport equation; consequently we cannot find with a certainty a direct physical meaning of the irreversibility of the process, it seems, however, that the situation closely resembles that in hydrodynamics.

There is no difficulty in writing down the principle of conservation of energy, basing on (2.31). It is deduced by introducing the density of the internal energy of the dislocation fluid ε, multiplying the first equation by u^i, the last by v^i, adding the results and integrating over a fixed volume. Now we can justify the omission of the last term in (2.24); in fact, this term multiplied by $\mathbf{v}$ is proportional to the expression $c_2^{-2}\nu\mathbf{v}\mathbf{v}\nabla\dot{\mathbf{u}}$ while from the first equation (2.31) we obtain a term $\nu\nabla\dot{\mathbf{u}}$; since $v^2/c_2^2 \ll 1$ the contribution of the considered term to the energy exchange between the elastic body and the dislocation fluid during any interval of time, is negligible. Therefore, it is consistent to neglect it.

The differential form of the law of energy conservation is the following:

$$\frac{\partial}{\partial t}\left[(\Pi+T)+\nu(\varepsilon+\tfrac{1}{2}m^{pq}v_p v_q)+\nu\varkappa_{pq}\left(\underset{H}{\sigma}^{pq}-\rho\frac{\partial u^p}{\partial t}v^q\right)\right]$$
$$+\nabla_r\left[-(\underset{H}{\sigma}^{pr}+\underset{D}{\sigma}^{pr})\frac{\partial u_p}{\partial t}+\varkappa_{pq}\underset{H}{\sigma}^{pq}\nu v^r+\nu\left(\varepsilon v^r-\frac{1}{\nu}\underset{K}{\sigma}^{pr}v_p+\tfrac{1}{2}m^{pq}v_p v_q v^r\right)\right]=0. \qquad (2.32)$$

Here Π and T are the potential and kinetic energies of the elastic body calculated by means of the displacement u_i. Thus, the density of the internal energy of the D_R medium is composed of the sum of the energies of its constituents completed by the interaction term

$$\nu\varkappa_{pq}\left(\underset{H}{\sigma}^{pq}-\rho\frac{\partial u^p}{\partial t}v^q\right).$$

Similarly, the energy flux vector is composed of the flux vectors of the constituents and the interaction term

$$\nu\varkappa_{pq}\underset{H}{\sigma}^{pq}v^r-\underset{D}{\sigma}^{pr}\frac{\partial u_p}{\partial t}.$$

To end the section consider the expression for the mechanical stress in the solid body; the term "solid body" is used intentionally, since the presence of the dislocations introduces a new term into the constitutive relation and the body is no longer elastic. Consider, therefore, a single small Somigliana dislocation; the permanent discontinuous displacement it generates, to within a rigid displacement of the whole body, has the form

$$\underset{\alpha}{\overset{p}{u}}_i = \underset{\alpha}{b}_i\underset{\alpha}{\eta}(s) \qquad (2.33)$$

where $\underset{\alpha}{\eta}(s)$ is the Heaviside function on the surface of the dislocation. The corresponding strain is obtained by differentiation:

$$\underset{\alpha}{\overset{p}{\varepsilon}}_{ij} = \nabla_{(i}\overset{p}{u}_{j)} = \underset{\alpha}{b}_{(i}\nabla_{j)}\underset{\alpha}{\eta}(s) = \underset{\alpha}{b}_{(i}\underset{\alpha}{n}_{j)}\underset{\alpha}{\delta}(s) \qquad (2.34)$$

where $\underset{\alpha}{\delta}(s)$ is the Dirac function; the latter can be represented as an integral over the surface $\underset{\alpha}{s}$ of the three-dimensional Dirac function. Taking into account that $\underset{\alpha}{s}$ is very small we have

$$\underset{\alpha}{\overset{p}{\varepsilon}}_{ij} = \int \mathrm{d}a\underset{\alpha}{b}_{(i}\underset{\alpha}{n}_{j)}\delta(\underset{\alpha}{\boldsymbol{\xi}}-\mathbf{x}) = \underset{\alpha}{\varkappa}_{(ij)}\delta(\underset{\alpha}{\boldsymbol{\xi}}-\mathbf{x}). \qquad (2.35)$$

Hence, the increment of the permanent strain during the time interval δt is given by the relation

$$\delta\underset{\alpha}{\overset{p}{\varepsilon}}_{ij} = \delta t\frac{\partial}{\partial t}[\underset{\alpha}{\varkappa}_{ij}\delta(\underset{\alpha}{\boldsymbol{\xi}}-\mathbf{x})] = -\delta t\nabla_p[\underset{\alpha}{\varkappa}_{(ij)}\underset{\alpha}{v}^p\delta(\underset{\alpha}{\boldsymbol{\xi}}-\mathbf{x})]. \qquad (2.36)$$

Setting now in the restricted theory $\varkappa_{ij} = \underset{\alpha}{U_i} n_j = \varkappa_{ij}$ we calculate the expectation value of the strain increment (2.30). We obtain

$$\delta \overset{p}{\varepsilon}_{ij}(\mathbf{x}, t) = \sum_{\alpha} \langle \delta \overset{p}{\underset{\alpha}{\varepsilon}}_{ij} \rangle = -\delta t \varkappa_{(ij)} \nabla_p [\nu(\mathbf{x}, t) v^p(\mathbf{x}, t)]. \tag{2.37}$$

Thus, the total permanent strain at instant t is given by the integral

$$\overset{p}{\varepsilon}_{ij}(\mathbf{x}, t) = -\varkappa_{(ij)} \nabla_p \int_{-\infty}^{t} \mathrm{d}\tau \nu(\mathbf{x}, t) v^p(\mathbf{x}, \tau). \tag{2.38}$$

Finally, making use of the continuity equation for the dislocation fluid (2.16) we have

$$\overset{p}{\varepsilon}_{ij}(\mathbf{x}, t) = \varkappa_{(ij)} \nu(\mathbf{x}, t). \tag{2.39}$$

Now, by definition of the average displacement, the total strain at point $\mathbf{x}$ and instant t is given by the usual relation $\varepsilon_{ij}(\mathbf{x}, t) = \nabla_{(i} u_{j)}(\mathbf{x}, t)$. Thus, the elastic strain connected with the stress $\underset{H}{\sigma}^{ij}$ by the Hooke law is the difference $\varepsilon_{ij} - \overset{p}{\varepsilon}_{ij}$ and for the mechanical stress we obtain the relation

$$\sigma_{ij} = \underset{H}{\sigma}_{ij} + (\lambda \delta_{ij} U_{(n)} + 2\mu \varkappa_{(ij)}) \nu. \tag{2.40}$$

Observe that in the base of tangenial dislocations

$$\sigma_p{}^p = (3\lambda + 2\mu) \nabla_p u^p.$$

The formulae (2.39) and (2.40) were deduced in different theories before (see e.g. [5]). Now, ν is involved in a system of differential equations with $\underset{H}{\sigma}^{ij}$ and $\partial^2 u_i / \partial t^2$ and, therefore, in general, the classical statement of the constitutive relation connecting the measurable (mechanical) stress σ^{ij} with the measurable strain $\varepsilon_{ij} = \nabla_{(i} u_{j)}$ can be constructed only *a posteriori*, after having solved the system (2.31) with the appropriate initial and boundary conditions. In this sense, no "plasticity condition" is assumed beforehand but it should follow from the theory itself. Furthermore, in general, σ_{ij} is a functional of the past. As mentioned before, even when the initial solid body is perfectly elastic, the dynamics of the discrete system is invariant with respect to the inversion of time and the basic model of the defect does not change in the course of the process, as is the case in our theory, in view of the generation of the discontinuity surfaces, the process is irreversible.

3. ONE-DIMENSIONAL MOTION OF TANGENTIAL DISLOCATIONS

This example will serve to demonstrate some properties of the discontinuity surfaces arising in a kind of a shear of an infinite space. Assume first that the dislocations are tangential, i.e. $\mathbf{U} \cdot \mathbf{n} = 0$; without loss of generality we can assume that $\mathbf{U} = U(1, 0, 0)$ and $\mathbf{n} = (0, 1, 0)$; further, assume that the motion occurs in the xy-plane, i.e. $u_3 = v_3 = 0$ and all quantities depend only on y and t. Then the system of equations (2.31) takes the form

$$\begin{gathered}\Box_1^2 u_2 = -\frac{1}{\rho}X_2, \qquad \frac{\partial v_1}{\partial t}+v_2\frac{\partial v_1}{\partial y} = 0 \\ \Box_2^2 u_1 + U\left[c_2^2\frac{\partial \nu}{\partial y}+\frac{\partial}{\partial t}(\nu v_2)\right] = -\frac{1}{\rho}X_1 \\ \frac{\partial \nu}{\partial t}+\frac{\partial}{\partial y}(\nu v_2) = 0 \\ \frac{\partial v_2}{\partial t}+v_2\frac{\partial v_2}{\partial y}+\frac{c^2}{\nu}\frac{\partial \nu}{\partial y}-\frac{\chi}{U}\frac{\partial^2 u_1}{\partial t^2} = \psi(y,t)\end{gathered} \tag{3.1}$$

where

$$\Box_{1,2}^2 = c_{1,2}^2\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial t^2}, \qquad \chi = 2\rho \overset{-1}{m}_{22}U^2, \qquad \psi(y,t) = -2\mu \overset{-1}{m}_{22}\frac{\partial^2 u_2}{\partial y^2}.$$

Thus, $u_2(y,t)$ can be found independently (e.g. $u_2 \equiv 0$ when $X_2 = 0$) and $v_1(y,t)$ can be determined after u_1, ν_1, v_2 have been found. Our system of equations therefore (3.1[3])–(3.1[5]) consists of three quasi-linear equations with three unknowns. As mentioned before, the theory of such equations is fairly well developed ([14–17]).

The characteristic velocities are the following

$$\lambda_{1,2} = \pm c_2 \tag{3.2}$$

$$\lambda_{3,4} = \frac{1}{1-\chi\nu}\left\{v_2 \pm c\sqrt{\left[\chi\nu\left(\frac{v_2^2}{c^2}-1\right)+1\right]}\right\} \tag{3.3}$$

and the hyperbolicity condition has the form

$$\chi\nu\left(\frac{v_2^2}{c^2}-1\right)+1 > 0. \tag{3.4}$$

In the case of vanishing $\chi\nu$ the propagation of weak waves in the two media becomes independent and we obtain the familiar results $\lambda_{1,2} = \pm c_2$, $\lambda_{3,4} = v_2 \pm c$. Observe that since $\chi\nu > 0$ the supersonic motion $v_2^2/c^2 > 1$ is always hyperbolic. We assume hereafter that (3.4) is satisfied, i.e. the Cauchy problem is well posed.

Now, it is known that no matter how continuous the initial conditions are, regular solutions of (3.1) do not exist globally and we have to consider weak solutions, i.e. in our case ν and v_2 will have finite jumps on certain surfaces, similarly to the first derivatives of u_1. These surfaces of strong discontinuities, in the absence of the body forces X_1, X_2, will be in our case the lines $y = \text{const.}$ in the xy-plane, and will be propagated with certain velocities different from (3.2) or (3.3). To investigate this propagation we derive in the usual manner the relations between the considered jumps, which we shall call the dynamic compatibility conditions (sometimes the name generalized Rankine–Hugoniot relations is used); taking into account that the kinematic compatibility condition ([7], Section 190) reads $[\partial u_1/\partial t] = (\partial u_1/\partial t)^+ - (\partial u_1/\partial t)^- = -c^*[\partial u_1/\partial y]$ where c^* is the velocity of propagation of the considered strong discontinuity, and that

$$\underset{H}{\sigma_{12}} = \mu\frac{\partial u_1}{\partial y}, \qquad \sigma_{12} = \underset{H}{\sigma_{12}}+\mu U \nu,$$

we obtain writing $\nu = \frac{1}{2}(\nu^+ + \nu^-)$, $\quad v_2 = \frac{1}{2}(v_2^+ + v_2^-)$,

$$\left(1 - \frac{c^{*2}}{c_2^2}\right)([\underset{H}{\sigma_{12}}] + \mu U[\nu]) = 0 \quad \text{or } [\sigma_{12}] = 0, \text{ since } c^* \ll c_2$$

$$(v_2 - c^*)[\nu] + \nu[v_2] = 0 \tag{3.5}$$

$$(v_2 - c^*)[v_2] + \frac{\gamma}{\gamma - 1}\left[\frac{\rho}{\nu}\right] + \chi(c^*)^2[\nu] = 0.$$

This is a system of homogeneous equations; equating to zero its determinant we obtain the expression for c^*:

$$c_{1,2}^* = \frac{1}{1 - \chi\nu}\left\{v_2 \pm c^+ \sqrt{\left[\chi\nu\frac{v_2^2}{(c^+)^2} + (1 - \chi\nu)\frac{1 + v_+^-}{2(\gamma - 1)}\,\frac{1 - (\nu_+^-)^{\gamma - 1}}{1 - \nu_+^-}\right]}\right\} \tag{3.6}$$

here $\nu_+^- = \nu^-/\nu^+$ and c^+ is the value of c on the "+" side of the discontinuity surface. For theorems concerning the problems of existence and uniqueness we refer the reader to the above mentioned references. A method of selecting physically meaningful solutions is to consider the parabolic system with $n^{ij} \neq 0$ and select the solutions obtained as $n^{ij} \to 0$.

Acknowledgements—The author is grateful to Dr. Z. Mossakowska, Dr. J. Krzeminski and Dr. Ł. Turski for the discussion of certain parts of the manuscript.

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APPENDIX A

STATISTICAL THEORY OF CONTINUOUS DISTRIBUTIONS OF VACANCIES

It has been conjectured by some authors (see e.g. [18]) that the fracture of certain types of materials can be explained by the concentration of vacancies. In this connection a statistical theory of continuous distributions of vacancies may be of interest. The basic equations can be derived as for the dislocations by changing the model as shown below. However, an examination of the derivation of (2.31) indicates that the same can be obtained directly from (2.31).

To construct a vacancy, [11], we assume that the surface of the dislocation $\varepsilon \underset{\alpha}{s}$ is a sphere of a very small radius ε; further we set $\mathbf{U} = U\mathbf{n}$ and consider U as a known constant. The required vacancy is then obtained by integrating over $\underset{\alpha}{s}$. Performing the integration in (2.31), in view of the smallness of ε we assume that all functions appearing in (2.31) are calculated at the center of the vacancy. Taking into account that

$$\int_{\underset{\alpha}{s}} \mathrm{d}a = 4\pi\varepsilon^2, \qquad \int_{\underset{\alpha}{s}} \mathrm{d}an_i n_j = \frac{4\pi\varepsilon^2}{3}\delta_{ij}$$

and dividing throughout by ε^2 we obtain

$$\begin{aligned} \mu\nabla^2 u_1 + (\lambda+\mu)\nabla_i\nabla_p u^p - \rho\frac{\partial^2 u_i}{\partial t^2} + \frac{1}{3}\mu U\left[K\nabla_i v + c_2^{-2}\frac{\partial}{\partial t}(vv_i)\right] &= -X_i \\ \frac{\partial v}{\partial t} + \nabla_p(vv^p) &= 0 \\ \frac{Dv_i}{Dt} + \frac{c^2}{v}\nabla_i v + \frac{1}{3}\frac{\mu U}{m}\left(K\nabla_i\nabla_p u^p - c_2^{-2}\frac{\partial^2 u_1}{\partial t^2}\right) &= 0 \end{aligned} \tag{A1}$$

where

$$K = \frac{3\lambda}{\mu} + 2, \qquad m = \mu c_2^{-5}\Delta^1 U^2[m_1 + m_2 + \tfrac{1}{3}(m_3 + m_4 + m_5)].$$

This is the required system.

APPENDIX B

ON THE TRANSFER OF MASS

In all our considerations it was tacitly assumed that there is no diffusion (transfer) of mass, i.e. that the defects possess no real mass. The case, however, may be different when we deal with defects such as interstitial atoms and vacancies. Then the following modification should be introduced to (2.31) and (A1).

First, the expression for the tensorial mass m^{ip} (1.4) should be replaced by the following:

$$m^{ip} = m\delta^{ip} + m^{ip} \tag{B1}$$

where m is the real mass of the dislocation. Secondly, the transfer of mass changes the Lamé equations. Thus, in a convected volume there are mass sources ([19, 7]) and the continuity equation for the linear elastic body has the form

$$\frac{\partial \rho'}{\partial t} + \rho \dot{u}_p{}^p = \rho s \tag{B2}$$

where ρ is the density in the natural state and hence the total density is $\rho + \rho'$; s denotes the mass sources per unit mass of the body. The presence of s influences the equations of motion in such a way that the body force is replaced by the expression

$$X_i - \rho s \dot{u}_i.$$

In our case $s = U_{(n)}\dot{v}$ and the first equation (2.31) takes the form

$$\begin{aligned} &\mu\nabla^2 u_i + (\lambda+\mu)\nabla_i\nabla_p u^p - \rho\frac{\partial^2 u_i}{\partial t^2} \\ &+ \mu\varkappa_{pq}\left[\sigma^{pqr}{}_i\nabla_r v + c_2^{-2}\delta_i^p\frac{\partial}{\partial t}(vv^q)\right] \\ &- \rho U_{(n)}\dot{v}u_i = -X_i. \end{aligned} \tag{B3}$$

The remaining equations of the system are of course unchanged. The corresponding equation (A1) is the following:

$$\mu\nabla^2 u_i + (\lambda+\mu)\nabla_i\nabla_p u^p - \rho\frac{\partial^2 u_i}{\partial t^2} + \frac{1}{3}\mu U_{(n)}\left[K\nabla_i v + c_2^{-2}\frac{\partial}{\partial t}(vv_i)\right] - \rho U_{(n)}\dot{v}\dot{u}_i = -X_i. \tag{B4}$$

It is to be borne in mind that in the case of vacancies $U_{(n)} < 0$.

(*Received* 26 *November* 1967)

Абстракт—Целью работы является статистический вывод уравнений, касающихся непреривного распределения дислокаций в линейной упругой среде. Автор исходит из системы инфинимитезимальных дислокаций Сомильяна, которые передвигаются в упругой среде, в согласии с законами динамики дискретных дислокаций. Путем введения классического фазного просранства с его уравнениями Лиувилля и движения и, предполагая ожидаемые значения, определяются, обычным способом, уравнения для плотности жидкости дислокации, и, ее скорости, а также средное поле упругости. В резулвтате получается сложная непрерывная среда D_R, которая является смесью материального упругого тела и жидкости дислокации. Система уравнений состоит из системы 7 квази-линейных дифференциальных уравнений в частных производных, которые являются гиперболическими при некоторых заданных условиях. Обсуждаются некоторые общие свойства системы. Представляется, детально, пример для указания некоторых свойств среды D_R. Оказываются, что в данном случае появляются ударные волны и плоскости скольжения. Доказывается коротко возможность построения таким образом "пластической" или "упруго-пластической" сред.